Asymptotic counting in dynamical systems

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Introduction

Question

Given a counting function which quantifies some measurable property of a geometric object, what are the asymptotics of such a function?

- Expectation: $\sim cx^d$, d is "dimension"
- Two Main Examples
  - Fatou Components of Rational Maps
  - Limit Sets of Schottky Groups

Dimension

Definition (Box-Counting Dimension)

Suppose that $N(\varepsilon)$ is the number of boxes of side length ε required to cover a set S. Then the dimension of S is defined as

$$\lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)}.$$

- unique d such that $N(1/x) \sim cx^d$ as $x \to \infty$.
- motivation for power law

Dimension Example: East Coast of Britain

Figure: East Coast of Britain, $d \approx 1.21$

Dimension Example: Cantor Set

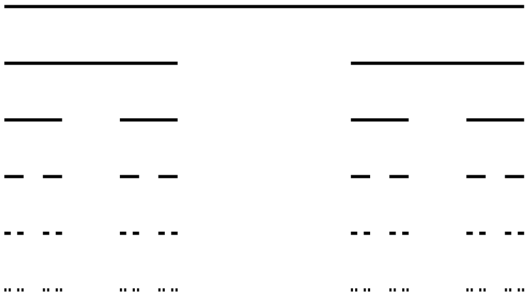


Figure: Cantor Set, $d = \log_3 2$

Julia Set and Fatou Components

Definition

The *filled-in Julia set* of a complex function f is defined as

$$K(f) = \{z \in \mathbb{C} : f^k(z) \nrightarrow \infty\}.$$

Definition

The *Julia Set* of f is defined as the boundary of the filled-in Julia set, i.e.

$$J(F) = \partial K(F).$$

Definition

A *Fatou component* of f is a connected component of $K(f)$.

Examples: Julia set of $f(z) = z^2 + c$

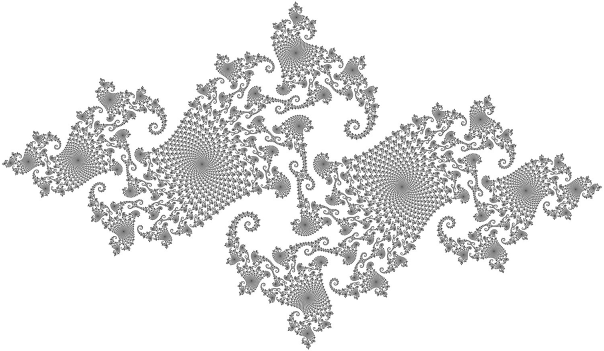


Figure: $c = -0.74543 + 0.11301i$

Examples: Julia set of $f(z) = z^2 + c$

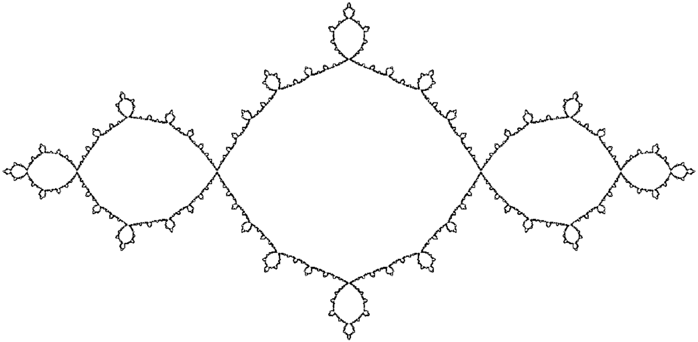


Figure: The Basilica ($c = -1$)

Main Theorem on Fatou Components

Definition

Given a set $X \subset \mathbb{C}$, define the *diameter* of X as

$$\mathrm{diam}(X) = \sup\{|x - y| : x, y \in X\}.$$

The Counting Function

For every function $f : \mathbb{C} \to \mathbb{C}$, we associate a counting function $N_f : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ where $N_f(x)$ is the number of Fatou components of f whose diameter is at least $1/x$.

Main Result on Fatou Components Cont.

Conjecture

Suppose $f(z) = z^2 + c$ has an infinite number of Fatou components. Then,

$$N_f(x) \sim c_f x^d$$

where d is the dimension of $J(f)$ and $c_f > 0$ is a constant.

- Numerically verified the theorem for $f(z) = z^2 - 1$.
- Proved in paper by M. Pollicott, M. Urbanski

Algorithm: Diameters of the Basilica

Escaping Criterion

Let $f(z) = z^2 + c$ be a complex quadratic function. Let $R = \frac{1+\sqrt{1+4|c|}}{2}$. If for some $n > 0$ we have $|f^n(z_0)| > R$, then $z_0 \notin K(f)$.

1. Construction
2. Distinguish Components
3. Computation of Diameter
4. Problem with (2): Bridges
5. $\sqrt{A}$ Counting Function

Figure: Bridge

Hyperbolic Geometry: The Poincare Disk Model

Definition

The *Poincare disk* is the unit disk $\mathbb{D}$ equipped with new notions of lines, distance, and angles.

- Geodesics are orthogonal circles!
- To calculate distance, we can use the formula

$$d(A,B) = \ln \frac{|AQ| \cdot |BP|}{|AP| \cdot |BQ|}$$

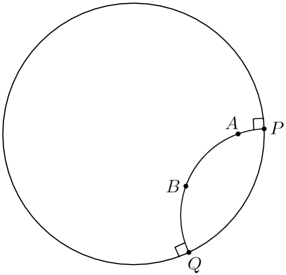


Figure: Geodesics

Angles in $\mathbb{D}$

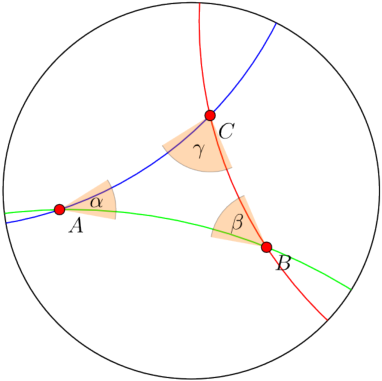


Figure: Angle between Geodesics

Isometries and Schottky Groups

Definition

An *isometry* is a map that preserves distances.

- Isometry Groups
- $$G = \left\{ h(z) = \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}} : |\alpha|^2 - |\beta|^2 = 1,\ \alpha, \beta \in \mathbb{D} \right\}.$$

Isometries and Schottky Groups (Contd)

Definition

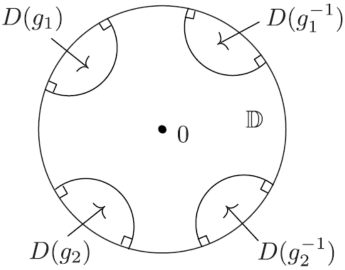
If g is an isometry which does fix z_0, then $D_{z_0}(g) = D(g)$ represents the closed half-plane in $\mathbb{D}$ bounded by the perpendicular bisector of the hyperbolic segment $[z_0, g(z_0)]$ containing $g(z_0)$.

Definition

A Schottky group of rank 2 is a subgroup of G generated by two isometries g_1, g_2 that satisfies

$$\overline{(D(g_1) \cup D(g_1^{-1}))} \cap \overline{(D(g_2) \cup D(g_2^{-1})} = \emptyset$$

Example: Schottky Groups



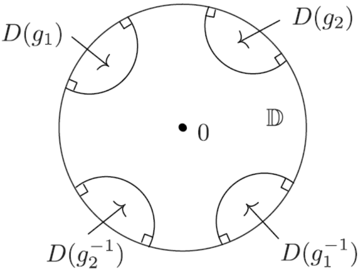


Figure: Induced Half Planes of the Generators

The Limit Set of a Schottky Group

Definition

The *limit set* of $L(\Gamma)$ of a isometry group Γ is defined by

$$L(\Gamma) = \overline{\Gamma z} \cap \partial \mathbb{D}$$

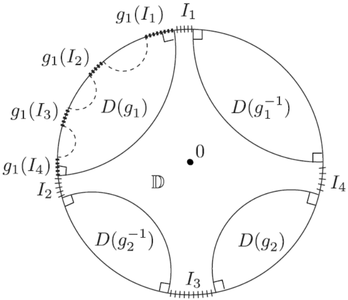


Figure: $L(S(g_1, g_2))$

Conjecture on the Limit Sets $S(g_1, g_2)$

Definition (Counting Function)

For a Schottky group $S(g_1, g_2)$, define $N(x, p) : \mathbb{R}_{>0} \times \mathbb{D} \to \mathbb{R}_{\geq 0}$ to be the number of intervals $I \in \partial\mathbb{D} \backslash L(S(g_1, g_2))$ such that the angle between the geodesics from the endpoints of I to p is at least $1/x$.

Conjecture

For every point $p \in \mathbb{D}$ and nontrivial Schottky group $S(g_1, g_2)$,

$$N(x, p) \sim cx^d$$

where d is the dimension of $L(S(g_1, g_2))$. The constant c depends on the point p.

Future Research

- Fatou Components
  - Generalize the theorem for different types of rational maps
  - Prove a similar theorem replacing diameters by $\sqrt{A}$.
- Limits Sets of Schottky Groups
  - Efficient Algorithm for limit sets of Schottky groups
  - verify and prove the conjecture on Schottky groups
  - Relationship between c and p
- Discover new counting functions which are asymptotic to cx^d.

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References

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Image Sources

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